Here $h = t_{n+1} - t_n$. The coefficients $\{c_i, a_{i,j}, b_j\}$ are given and they define the numerical method. The function F of (5.14), defining a one-step method, is defined implicitly through the formulas (5.22)-(5.23).

More succinctly, we can write the formulas as

$$z_i = y_n + h \sum_{j=1}^{i-1} a_{i,j} f(t_n + c_j h, z_j)\,, \qquad i = 1, \ldots, s, \tag{5.24}$$

$$y_{n+1} = y_n + h \sum_{j=1}^{s} b_j f(t_n + c_j h, z_j)\,. \tag{5.25}$$

The coefficients are often displayed in a table called a *Butcher tableau* (after J. C. Butcher):

$$\begin{array}{c|ccccc} 0 = c_1 & & & & & \\ c_2 & a_{2,1} & & & & \\ c_3 & a_{3,1} & a_{3,2} & & & \\ \vdots & \vdots & & \ddots & & \\ c_s & a_{s,1} & a_{s,2} & \cdots & a_{s,s-1} & \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array} \tag{5.26}$$

The coefficients $\{c_i\}$ and $\{a_{i,j}\}$ are usually assumed to satisfy the conditions

$$\sum_{j=1}^{i-1} a_{i,j} = c_i, \qquad i = 2, \ldots, s. \tag{5.27}$$

Example 5.3 We give two examples of well-known Runge–Kutta methods.

- The method (5.20) has the Butcher tableau

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

- A popular classical method is the following fourth-order procedure.

$$\begin{aligned} z_1 &= y_n, \\ z_2 &= y_n + \tfrac{1}{2} h\, f\left(t_n, z_1\right), \\ z_3 &= y_n + \tfrac{1}{2} h\, f\left(t_n + \tfrac{1}{2} h, z_2\right), \\ z_4 &= y_n + h\, f\left(t_n + \tfrac{1}{2} h, z_3\right), \end{aligned}$$

$$\begin{aligned} y_{n+1} = y_n + \tfrac{1}{6} h \big[& f\left(t_n, z_1\right) + 2 f\left(t_n + \tfrac{1}{2} h, z_2\right) \\ & + 2 f\left(t_n + \tfrac{1}{2} h, z_3\right) + f\left(t_n + h, z_4\right) \big]\,. \end{aligned} \tag{5.28}$$

The Butcher tableau is

$$
\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
1/2 & 0 & 1/2 & & \\
1 & 0 & 0 & 1 & \\
\hline
 & 1/6 & 1/3 & 1/3 & 1/6
\end{array}
\tag{5.29}
$$

Following an extended calculation modeled on that in (5.18), we can show $T_{n+1} = \mathcal{O}(h^5)$.

When the differential equation is simply $Y'(t) = f(t)$ with no dependence of f on Y, this method reduces to Simpson's rule for numerical integration on $[t_n, t_{n+1}]$. The method (5.28) can be easily implemented using a computer or a programmable hand calculator, and it is generally quite accurate. A numerical example is given at the end of the next section. ■

5.3 CONVERGENCE, STABILITY, AND ASYMPTOTIC ERROR

We want to examine the convergence of the one-step method

$$y_{n+1} = y_n + hF(t_n, y_n; h), \qquad n \geq 0, \qquad y_0 = Y_0 \tag{5.30}$$

to the solution $Y(t)$ of the initial value problem

$$
\begin{aligned}
&Y'(t) = f(t, Y(t)), \qquad t_0 \leq t \leq b,\\
&Y(t_0) = Y_0.
\end{aligned}
\tag{5.31}
$$

Using the truncation error of (5.16) for the true solution Y, we introduce

$$\tau_n(Y) = \frac{1}{h} T_{n+1}(Y).$$

In order to show convergence of (5.30), we need to have $\tau_n(Y) \to 0$ as $h \to 0$. Since

$$\tau_n(Y) = \frac{Y(t_{n+1}) - Y(t_n)}{h} - F(t_n, Y(t_n), h; f), \tag{5.32}$$

we require that

$$F(t, Y(t), h; f) \to Y'(t) = f(t, Y(t)) \qquad \text{as } h \to 0.$$

Accordingly, define

$$\delta(h) = \sup_{\substack{t_0 \leq t \leq b \\ -\infty < y < \infty}} |f(t, y) - F(t, y, h; f)|, \tag{5.33}$$

and assume

$$\delta(h) \to 0 \qquad \text{as } h \to 0. \tag{5.34}$$