and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,\mathrm{d}t \quad (n = 1, 2, 3, \dots)$$

$$= \frac{2}{\pi}\int_0^{\pi} t^2\cos nt\,\mathrm{d}t = \frac{2}{\pi}\left[\frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{2\pi}{n^2}\cos n\pi\right) = \frac{4}{n^2}(-1)^n$$

since $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$. Thus the Fourier series expansion of $f(t) = t^2$ is

$$f(t) = \tfrac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nt \tag{7.22}$$

or, writing out the first few terms,

$$f(t) = \tfrac{1}{3}\pi^2 - 4\cos t + \cos 2t - \tfrac{4}{9}\cos 3t + \dots$$

7.2.5 Linearity property

The linearity property as applied to Fourier series may be stated in the form of the following theorem.

Theorem 7.1 If $f(t) = lg(t) + mh(t)$, where $g(t)$ and $h(t)$ are periodic functions of period T and l and m are arbitrary constants, then $f(t)$ has a Fourier series expansion in which the coefficients are the sums of the coefficients in the Fourier series expansions of $g(t)$ and $h(t)$ multiplied by l and m respectively.

Proof Clearly $f(t)$ is periodic with period T. If the Fourier series expansions of $g(t)$ and $h(t)$ are

$$g(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos n\omega t + \sum_{n=1}^{\infty} b_n\sin n\omega t$$

$$h(t) = \tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n\cos n\omega t + \sum_{n=1}^{\infty} \beta_n\sin n\omega t$$

then, using (7.4) and (7.5), the Fourier coefficients in the expansion of $f(t)$ are

$$A_n = \frac{2}{T}\int_d^{d+T} f(t)\cos n\omega t\,\mathrm{d}t = \frac{2}{T}\int_d^{d+T}[lg(t) + mh(t)]\cos n\omega t\,\mathrm{d}t$$

$$= \frac{2l}{T}\int_d^{d+T} g(t)\cos n\omega t\,\mathrm{d}t + \frac{2m}{T}\int_d^{d+T} h(t)\cos n\omega t\,\mathrm{d}t = la_n + m\alpha_n$$