```
> restart;
> f:=(a+b+b/a);
```

$$f := a + b + \frac{b}{a}$$

```
> fa:=diff(f, a):
> faa:=diff(f, a, a):
> faaa:=diff(f, a,a,a):
> fb:=diff(f, b):
> fbb:=diff(f, b,b):
> fab:=diff(f, a, b):
> fabb:=diff(f, a, b,b):
> fbbb:=diff(f, b,b,b):
> faab:=diff(f, a,a,b):
> rho:=f*fa/a:
> rho:=simplify(rho);
```

$$\rho := \frac{(a^2 + b\,a + b)\,(a^2 - b)}{a^4}$$

```
> rho0:=f*fbb+fb*fb:
> rho0:=simplify(rho0);
```

$$\rho 0 := \frac{(a+1)^2}{a^2}$$

```
> rho1:=(f*fab+fa*fb)/a:
> rho1:=simplify(rho1);
```

$$\rho 1 := \frac{-2\,b\,a - 2\,b + a^3}{a^4}$$

```
> rho2:=((f*faa+fa^2)*a-f*fa)/a^3:
> rho2:=simplify(rho2);
```

$$\rho 2 := -\frac{b\,(-3\,b\,a - 4\,b + a^3)}{a^6}$$

```
> g_ij:=rho*a_ij + rho0*b_i*b_j+rho1*(b_i*y_j+b_j*y_i)+rho2*y_i*y_j;
```

$$g\_ij := \frac{(a^2 + b\,a + b)\,(a^2 - b)\,a\_ij}{a^4} + \frac{(a+1)^2\,b\_i\,b\_j}{a^2} + \frac{(-2\,b\,a - 2\,b + a^3)\,(b\_i\,y\_j + b\_j\,y\_i)}{a^4} - \frac{b\,(-3\,b\,a - 4\,b + a^3)\,y\_i\,y\_j}{a^6}$$

```
> epsilon:=rho1/rho2:
> epsilon:=simplify(epsilon);
```

$$\varepsilon := -\frac{(-2\,b\,a - 2\,b + a^3)\,a^2}{b\,(-3\,b\,a - 4\,b + a^3)}$$

```
> delta:=(rho0-epsilon^2*rho2)/rho:
```

```
> delta:=simplify(delta);
```

$$\delta := \frac{(a^3 - 3\,b\,a - 3\,b)\,a^3}{(a^2 - b)\,b\,(-3\,b\,a - 4\,b + a^3)}$$

```
> mu:=rho2/rho:
> mu:=simplify(mu);
```

$$\mu := -\frac{b\,(-3\,b\,a - 4\,b + a^3)}{a^2\,(a^2 + b\,a + b)\,(a^2 - b)}$$

```
> u:=1+mu*(a^2+epsilon*b):
> v:=delta*b+mu*epsilon*(a^2+epsilon*b):

> u:=simplify(u);
```

$$u := \frac{a^4 + b\,a^3 + b^2}{(a^2 + b\,a + b)\,(a^2 - b)}$$

```
> v:=simplify(v);
```

$$v := \frac{(a^3 + b\,a^2 - b)\,a^2}{(a^2 + b\,a + b)\,(a^2 - b)}$$

```
> h_ij:=aa*a_ij+bb*b_i*b_j+cc*(b_i*y_j+b_j*y_i)+dd*y_i*y_j:

> aa:=simplify(rho);
```

$$aa := \frac{(a^2 + b\,a + b)\,(a^2 - b)}{a^4}$$

```
> bb:=simplify(rho0-rho^2*v^2/f^2);
```

$$bb := \frac{6\,b\,a^3 + 4\,b^2\,a^2 + 8\,b^2\,a + 2\,a^4 + 8\,b\,a^2 + 4\,b^2 + a^3 + 2\,b\,a}{a\,(a^2 + b\,a + b)^2}$$

```
> cc:=simplify(rho1-rho^2*v/f^2);
```

$$cc := -\frac{(b\,a^2 + 4\,b\,a + a^2 + 3\,b)\,b}{a^4\,(a^2 + b\,a + b)}$$

```
> dd:=simplify(rho2-rho^2/f^2);
```

$$dd := -\frac{-3\,b^2\,a - 3\,b^2 + b\,a^3 + a^4 - 2\,b\,a^2}{a^6}$$

```
> h_ik:=aa*a_ik+bb*b_i*b_k+cc*(b_i*y_k+b_k*y_i)+dd*y_i*y_k:
> h_jk:=aa*a_jk+bb*b_j*b_k+cc*(b_j*y_k+b_k*y_j)+dd*y_j*y_k:


> tau:=delta/(1+delta*b2):
> tau:=simplify(tau);
```

$$\tau := \frac{(a^3 - 3\,b\,a - 3\,b)\,a^3}{-4\,b^2\,a^3 - 4\,b^2\,a^2 + b\,a^5 + 3\,b^3\,a + 4\,b^3 + a^6\,b2 - 3\,a^4\,b2\,b - 3\,a^3\,b2\,b}$$

```
> lambda:=(epsilon-delta*b)/(1+delta*b2):
> lambda:=simplify(lambda);
```

$$\lambda := -\frac{a^2\,(-3\,b\,a^3 - b^2\,a - 2\,b\,a^2 + 2\,b^2 + a^5 + b\,a^4 - 3\,b^2\,a^2)}{-4\,b^2\,a^3 - 4\,b^2\,a^2 + b\,a^5 + 3\,b^3\,a + 4\,b^3 + a^6\,b2 - 3\,a^4\,b2\,b - 3\,a^3\,b2\,b}$$

```
> Y2:=a^2+(lambda+epsilon)*b+lambda*epsilon*b2:
> eta:=mu/(1+Y2*mu):
> eta:=simplify(eta);
```

$$\eta := -(-4\,b^2\,a^3 - 4\,b^2\,a^2 + b\,a^5 + 3\,b^3\,a + 4\,b^3 + a^6\,b2 - 3\,a^4\,b2\,b - 3\,a^3\,b2\,b)/(a^2(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3))$$

```
> gij:=rho^(-1)*(aij-tau*bi*bj-eta*(yi+lambda*bi)*(yj+lambda*bj));
```

$$gij := a^4(aij - \frac{(a^3 - 3\,b\,a - 3\,b)\,a^3\,bi\,bj}{\%1} + \%1 (yi - \frac{a^2\,(-3\,b\,a^3 - b^2\,a - 2\,b\,a^2 + 2\,b^2 + a^5 + b\,a^4 - 3\,b^2\,a^2)\,bi}{\%1}) (yj - \frac{a^2\,(-3\,b\,a^3 - b^2\,a - 2\,b\,a^2 + 2\,b^2 + a^5 + b\,a^4 - 3\,b^2\,a^2)\,bj}{\%1})/(a^2(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3)))/((a^2 + b\,a + b)\,(a^2 - b))$$

$$\%1 := -4\,b^2\,a^3 - 4\,b^2\,a^2 + b\,a^5 + 3\,b^3\,a + 4\,b^3 + a^6\,b2 - 3\,a^4\,b2\,b - 3\,a^3\,b2\,b$$

```
> De:=simplify(rho^n*(1+delta*b2)*(1+mu*Y2));
```

$$De := (a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3)(\frac{(a^2 + b\,a + b)\,(a^2 - b)}{a^4})^n \Big/ ((a^2 - b)^2\,(a^2 + b\,a + b))$$

```
> detg_ij:=De*deta_ij;
```

$$detg\_ij := (a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3)(\frac{(a^2 + b\,a + b)\,(a^2 - b)}{a^4})^n\,deta\_ij \Big/ ((a^2 - b)^2\,(a^2 + b\,a + b))$$

Compute the Cartan torsion C_ijk

```
> C_1:=simplify(a^2*f*faaa-3*a*f*faa+3*a^2*fa*faa-3*a*fa^2+3*f*fa);
```

$$C\_1 := \frac{3\,b\,(-5\,b\,a - 8\,b + a^3)}{a^3}$$

```
> C_2:=simplify(2*a^3*fa*fab+a^3*f*faab-a^2*fa*fb+a^3*faa*fb-a^2*f*fab)
> ;
```

$$C\_2 := -\frac{-8\,b - 6\,b\,a + a^3}{a}$$

```
> C_3:=simplify(a^4*f*fabb+2*a^4*fb*fab+a^4*fa*fbb);
```

$$C\_3 := -2\,a\,(a+1)$$

```
> C_4:=simplify(a^5*f*fbbb+3*a^5*fb*fbb);
```

$$C\_4 := 0$$

```
> C_5:=simplify(a^3*fa^2+a^3*f*faa-a^2*f*fa);
```

$$C\_5 := -\frac{b\,(-3\,b\,a - 4\,b + a^3)}{a}$$

```
> C_6:=simplify(a^4*f*fab+a^4*fa*fb);
```

$$C\_6 := a\,(-2\,b\,a - 2\,b + a^3)$$

```
> C_ijk:=(C_1*y_i*y_j*y_k+C_2*(b_i*y_j*y_k+y_i*b_j*y_k+y_i*y_j*b_k)+C_3
> *(y_i*b_j*b_k+b_i*y_j*b_k+b_i*b_j*y_k)+C_4*b_i*b_j*b_k+C_5*(a_jk*y_i+a
> _ik*y_j+a_ij*y_k)+C_6*(a_jk*b_i+a_ik*b_j+a_ij*b_k))/(2*a^5);
```

$$\begin{aligned} C\_ijk := (&\frac{3\,b\,(-5\,b\,a - 8\,b + a^3)\,y\_i\,y\_j\,y\_k}{a^3} \\ &- \frac{(-8\,b - 6\,b\,a + a^3)\,(b\_i\,y\_j\,y\_k + y\_i\,b\_j\,y\_k + y\_i\,y\_j\,b\_k)}{a} \\ &- 2\,a\,(a+1)\,(y\_i\,b\_j\,b\_k + b\_i\,y\_j\,b\_k + b\_i\,b\_j\,y\_k) \\ &- \frac{b\,(-3\,b\,a - 4\,b + a^3)\,(a\_jk\,y\_i + a\_ik\,y\_j + a\_ij\,y\_k)}{a} \\ &+ a\,(-2\,b\,a - 2\,b + a^3)\,(a\_jk\,b\_i + a\_ik\,b\_j + a\_ij\,b\_k))/(2\,a^5) \end{aligned}$$

Compute the mean Cartan torsion

```
> I_i:=(M1*y_i+M2*b_i)/(2*a^5*rho):
```

```
> M1:=a^2*C_1+2*b*C_2+b2*C_3+(n+2)*C_5-tau*C_1*b^2-2*tau*C_2*b*b2-tau*C
> _3*b2^2-tau*C_5*b2-eta*C_1*(a^2+lambda*b)^2-2*eta*C_2*(b+lambda*b2)*(a
> ^2+lambda*b)-eta*C_3*(b+lambda*b2)^2-2*eta*C_5*(a^2+lambda*b)-eta*C_5*
> (a^2+2*lambda*b+lambda^2*b2)-eta*C_6*2*(b+lambda*b2):
> M1:=simplify(M1);
```

$$M1 := -(8\,b\,a^7\,b2 - 2\,b\,a^6\,b2 + 12\,b^2\,a^5\,b2 + 2\,b^2\,a^4\,b2 + 2\,b^2\,a^7\,b2 + 14\,b^2\,a^6\,b2 + 6\,b^3\,a^5\,b2 + 10\,b^3\,a^4\,b2 + 2\,a^3\,b2\,b^3 + 4\,a^8\,b2\,b - 2\,a^2\,b2\,b^3 - 9\,b^3\,a^5 - 10\,b^4\,a^3 - 4\,a^7\,b^2 + 2\,a^9\,b2 - 14\,a^4\,b^4 - 10\,b^3\,a^6 + 2\,b^2\,a^8 - 6\,b^5\,a^2 - 2\,b^5\,a - 6\,b^4\,a^5 - 3\,b^5\,a^3 + a^7\,b^3 + 2\,a^8\,b2 + a^9\,b - 12\,n\,b^2\,a^5\,b2 - 4\,n\,b^2\,a^4\,b2 + n\,a^7\,b2\,b - 4\,n\,b^2\,a^6 - 6\,n\,b^3\,a^5 + 4\,n\,b^3\,a^4 + 20\,n\,b^4\,a^3 + 4\,n\,b^4\,a^2 - 4\,n\,a^7\,b^2 + 2\,a^8\,b\,n\,b2 + 2\,a^7\,n\,b2\,b^2 - 5\,a^6\,n\,b2\,b^2 - 6\,a^5\,n\,b^3\,b2 - 11\,a^4\,n\,b^3\,b2 - a^3\,n\,b^3\,b2 + 4\,a^2\,n\,b^3\,b2 - 4\,n\,b^5 + 7\,a^4\,n\,b^4 + 9\,a^3\,n\,b^5 + 15\,a^2\,n\,b^5 + a\,n\,b^5 + a^9\,n\,b + 2\,a^8\,n\,b^2 + a^7\,n\,b^3 - 10\,a^6\,n\,b^3 - 6\,a^5\,n\,b^4)/(a(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3))$$

```
> M2:=C_2*a^2+2*C_3*b+C_4*b2+(n+2)*C_6-tau*C_2*b^2-2*tau*C_3*b*b2-tau*C
> _4*b2^2-2*tau*C_5*b-3*tau*C_6*b2-eta*C_2*(a^2+lambda*b)^2-2*eta*C_3*(b
> +lambda*b2)*(a^2+lambda*b)-eta*C_4*(b+lambda*b2)^2-2*eta*C_5*lambda*(a
> ^2+lambda*b)-2*eta*C_6*lambda*(b+lambda*b2)-eta*C_6*(a^2+2*b*lambda+la
> mbda^2*b2):
> M2:=simplify(M2);
```

$$M2 := (2\,a^4\,b2\,b + 10\,a^5\,b2\,b + 8\,b\,a^6\,b2 + 4\,b^2\,a^5\,b2 + 6\,b^2\,a^4\,b2 - 6\,b^2\,a^5 - 4\,b^3\,a^3 - 3\,a^7\,b - 8\,a^4\,b^3 - 8\,b^2\,a^6 - 5\,b^3\,a^5 + 2\,b\,a^8 + a^7\,b^2 + a^9 + 2\,a^7\,b2 - 2\,a^2\,b2\,b^2 - 3\,n\,b\,a^6\,b2 - 7\,n\,b\,a^5\,b2 - 4\,n\,b^2\,a^5\,b2 - 6\,n\,b^2\,a^4\,b2 - 2\,n\,b\,a^4\,b2 + 2\,n\,b^2\,a^2\,b2 + 2\,n\,a^7\,b2\,b - 2\,n\,b^4 + n\,a^9 - 3\,n\,b\,a^7 - 8\,n\,b^2\,a^6 - 3\,n\,b^2\,a^5 - 5\,n\,b^3\,a^5 + 5\,n\,b^3\,a^4 + 6\,n\,b^4\,a^3 + 11\,n\,b^3\,a^3 + 8\,n\,b^4\,a^2 - 2\,n\,b\,a^6 + 2\,n\,b^2\,a^4 + 2\,n\,b^3\,a^2 + 2\,n\,a^8\,b2 + 2\,n\,a^8\,b + n\,a^7\,b2 + n\,a^7\,b^2)a/(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3)$$

```
> I_i:=(M1*y_i+M2*b_i)/(2*a^5*rho):
> I_j:=(M1*y_j+M2*b_j)/(2*a^5*rho):
> I_k:=(M1*y_k+M2*b_k)/(2*a^5*rho):
```

C-reducibility

```
> CI:=simplify(C_ijk-(h_jk*I_i+h_ik*I_j+h_ij*I_k)/(n+1)):
```

Compute the coefficients of CI. In general, they are not equal to zero, except for Randers metric f = a+b and Kropina metric f = a^2/b. You can change f in the first line to verif this fact.

```
> D_1:=simplify(diff(CI, y_i, y_j, y_k));
```

$$D\_1 := -3(11\,b^2\,a^7\,b2 + 16\,b^2\,a^6\,b2 + b\,a^9\,b2 - 53\,b^3\,a^5\,b2 - 24\,b^3\,a^4\,b2 - 8\,a^8\,b2\,b$$
$$- 16\,b^3\,a^6 + 8\,b^2\,a^8 + 16\,b^5\,a^2 - 33\,b^4\,a^5 + 63\,b^5\,a^3 + 9\,a^7\,b^3 + 2\,a^{10}\,b2 - 8\,b^6$$
$$+ 8\,n\,b^4\,a^5\,b2 + 16\,n\,b^3\,a^6\,b2 + 12\,n\,b^4\,a^4\,b2 + 12\,b^3\,a^7\,b2 - 20\,b^3\,a^6\,b2$$
$$+ 13\,b^2\,a^8\,b2 - 28\,b^4\,a^5\,b2 - 51\,b^4\,a^4\,b2 - 9\,b^4\,a^3\,b2 + 2\,b\,a^{10}\,b2 + 14\,a^2\,b2\,b^4$$
$$- 9\,b^4\,a^6 + 18\,b^5\,a^4 + 47\,b^6\,a^2 + 9\,b^6\,a + 2\,a^{11}\,b2 - 8\,b^5\,a^5 + 24\,b^6\,a^3 - 12\,n\,b^6\,a^3$$
$$- 16\,n\,b^6\,a^2 + 4\,n\,b^5\,a^5 + 12\,n\,b^4\,a^6 - 24\,n\,b^5\,a^4 + 8\,a^7\,n\,b2\,b^2 + 4\,a^6\,n\,b2\,b^2$$
$$+ 16\,a^5\,n\,b^3\,b2 + 4\,n\,b^6 - 8\,a^4\,n\,b^4 - 24\,a^3\,n\,b^5 + 4\,a^8\,n\,b^2 + 12\,a^7\,n\,b^3$$
$$- 12\,a^5\,n\,b^4 - 4\,a^2\,n\,b^4\,b2)/(2\,(1+n)\,(a^2-b)(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2$$
$$+ 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3)a^8)$$

```
> D_2:=simplify(diff(CI, b_i, y_j, y_k));
```

$$D\_2 := -(-36\,n\,b^3\,a^9\,b2 - 52\,n\,b^3\,a^8\,b2 - 12\,n\,b^2\,a^{11}\,b2 - 88\,n\,b^4\,a^7\,b2 - 150\,n\,b^4\,a^6\,b2$$
$$- 26\,n\,b^3\,a^7\,b2 - 46\,n\,b^2\,a^{10}\,b2 - 90\,n\,b^4\,a^5\,b2 - 16\,n\,b^4\,a^8\,b2 - 8\,n\,b^3\,a^{10}\,b2$$
$$- 66\,n\,b^5\,a^6\,b2 - 116\,n\,b^5\,a^5\,b2 - 60\,n\,b^5\,a^4\,b2 + 24\,n\,b^5\,a^3\,b2 - 38\,n\,b^2\,a^9\,b2$$
$$- 12\,n\,b^4\,a^4\,b2 + 18\,b^2\,a^9\,b2 + 160\,b^3\,a^7\,b2 + 16\,b^3\,a^6\,b2 + 12\,b^2\,a^8\,b2$$
$$- 26\,b\,a^{11}\,b2 + 167\,b^4\,a^5\,b2 + 4\,b^4\,a^4\,b2 - 6\,b\,a^{10}\,b2 + 421\,b^4\,a^6\,b2$$
$$+ 181\,b^3\,a^8\,b2 - 63\,b^2\,a^{10}\,b2 + 102\,b^5\,a^4\,b2 - 30\,b^5\,a^3\,b2 - 45\,b^3\,a^{10}\,b2$$
$$- 14\,b^3\,a^9\,b2 + 6\,b^2\,a^{12}\,b2 + 63\,b^4\,a^8\,b2 + 335\,b^4\,a^7\,b2 - 49\,b^2\,a^{11}\,b2$$
$$+ 156\,b^5\,a^6\,b2 + 226\,b^5\,a^5\,b2 - 19\,b\,a^{12}\,b2 + 24\,b^3\,a^8 - 24\,b^5\,a^4 - 170\,b^5\,a^5$$
$$- 133\,b^6\,a^3 + 6\,a^7\,b^4 + 34\,b^3\,a^9 + 2\,b^7\,a - 25\,b^2\,a^{11} + 51\,b^4\,a^8 + 8\,b^7 - 7\,b^3\,a^{10}$$
$$- 219\,b^5\,a^6 - 363\,b^6\,a^4 - 8\,a^{12}\,b^2 - 52\,a^7\,b^5 + 33\,a^9\,b^4 - 325\,a^5\,b^6 - 150\,b^7\,a^3$$
$$- 78\,b^7\,a^2 - 10\,b^3\,a^{11} + 23\,b^5\,a^8 - 4\,b^4\,a^{10} - 97\,b^6\,a^6 - 102\,b^7\,a^4 - 24\,a^5\,b^7$$
$$- 8\,a^{12}\,b - a^{13}\,b2 - 22\,n\,b^7 + 6\,a^{13}\,b\,b2 + 2\,b^3\,a^{11}\,b2 - 14\,b^4\,a^9\,b2 + 36\,b^5\,a^7\,b2$$
$$- 26\,a^2\,b2\,b^5 + 134\,n\,b^6\,a^3 + 12\,n\,b^6\,a^2 + 48\,n\,b^5\,a^5 - 24\,n\,b^7\,a - 8\,n\,a^{12}\,b^2$$
$$- 24\,n\,a^{11}\,b^2 - 24\,n\,a^{10}\,b^3 + 10\,n\,a^9\,b^3 + 22\,n\,b^4\,a^8 + 2\,n\,b^4\,a^7 - 2\,n\,b^4\,a^9$$
$$- 10\,n\,b^3\,a^{11} - 10\,n\,b^5\,a^7 + 10\,n\,b^5\,a^6 - 2\,n\,b^5\,a^8 - 4\,n\,b^4\,a^{10} + 112\,n\,b^6\,a^5$$
$$+ 230\,n\,b^6\,a^4 + 96\,n\,b^7\,a^4 + 162\,n\,b^7\,a^3 + 82\,n\,b^7\,a^2 + 6\,n\,b^3\,a^8 - 8\,n\,b^4\,a^6$$
$$+ 22\,n\,b^5\,a^4 - 6\,n\,b^6\,a^7 + 18\,n\,b^7\,a^5 + 22\,n\,b^5\,a^2\,b2 - 4\,n\,a^{12}\,b2\,b - 12\,n\,b^5\,a^7\,b2$$
$$+ 2\,a^{14}\,b2 - 2\,a^{13}\,b - 4\,n\,b^2\,a^{10} - 4\,a^8\,n\,b2\,b^2 - 2\,n\,a^{13}\,b - 6\,n\,a^{12}\,b$$
$$- 14\,a^{11}\,b\,n\,b2 - 6\,a^{10}\,b\,n\,b2)/(2\,a^6\,(1+n)\,(a^2-b)(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4$$
$$+ a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a$$
$$+ b^3)(a^2 + b\,a + b)^2)$$

```
> D_3:=simplify(diff(CI, y_i, b_j, b_k));
```

$$
\begin{aligned}
D\_3 := -(&126\,n\,b^3\,a^9\,b2 + 191\,n\,b^3\,a^8\,b2 + 16\,n\,b^2\,a^{11}\,b2 + 374\,n\,b^4\,a^7\,b2 \\
&+ 226\,n\,b^4\,a^6\,b2 + 95\,n\,b^3\,a^7\,b2 + 33\,n\,b^2\,a^{10}\,b2 + 30\,n\,b^4\,a^5\,b2 + 232\,n\,b^4\,a^8\,b2 \\
&+ 6\,n\,b^3\,a^{10}\,b2 + 202\,n\,b^5\,a^6\,b2 + 120\,n\,b^5\,a^5\,b2 - 12\,n\,b^5\,a^4\,b2 - 38\,n\,b^5\,a^3\,b2 \\
&+ 12\,n\,b^3\,a^6\,b2 + 24\,n\,b^2\,a^9\,b2 - 10\,n\,b^4\,a^4\,b2 - 60\,b^2\,a^9\,b2 - 166\,b^3\,a^7\,b2 \\
&- 18\,b^3\,a^6\,b2 - 6\,b^2\,a^8\,b2 - 10\,b\,a^{11}\,b2 - 70\,b^4\,a^5\,b2 + 10\,b^4\,a^4\,b2 - 380\,b^4\,a^6\,b2 \\
&- 406\,b^3\,a^8\,b2 - 122\,b^2\,a^{10}\,b2 - 8\,b^5\,a^4\,b2 + 42\,b^5\,a^3\,b2 - 160\,b^3\,a^{10}\,b2 \\
&- 406\,b^3\,a^9\,b2 - 12\,b^2\,a^{12}\,b2 - 458\,b^4\,a^8\,b2 - 632\,b^4\,a^7\,b2 - 82\,b^2\,a^{11}\,b2 \\
&- 258\,b^5\,a^6\,b2 - 172\,b^5\,a^5\,b2 - 16\,b\,a^{12}\,b2 + 6\,b^4\,a^6 - 6\,b^3\,a^8 + 6\,b^5\,a^4 - 6\,b^2\,a^{10} \\
&- 2\,b^6\,a^2 + 126\,b^5\,a^5 + 38\,b^6\,a^3 + 96\,a^7\,b^4 + 5\,b^3\,a^9 - 6\,b^7\,a - 6\,b^2\,a^{11} + 234\,b^4\,a^8 \\
&- 2\,b^7 + 12\,b^3\,a^{10} + 434\,b^5\,a^6 + 230\,b^6\,a^4 + 4\,a^{12}\,b^2 + 577\,a^7\,b^5 + 180\,a^9\,b^4 \\
&+ 470\,a^5\,b^6 + 68\,b^7\,a^3 + 10\,b^7\,a^2 - 7\,b^3\,a^{11} + 322\,b^5\,a^8 + 26\,b^4\,a^{10} + 434\,b^6\,a^6 \\
&+ 110\,b^7\,a^4 + 8\,b^2\,a^{13} - 4\,a^{12}\,b^3 + 8\,a^{14}\,b + 180\,a^7\,b^6 + 56\,a^9\,b^5 + 74\,a^5\,b^7 \\
&- 10\,b^4\,a^{11} + 26\,b^6\,a^8 - 4\,b^5\,a^{10} + 18\,b^7\,a^6 + 2\,a^{12}\,b + 10\,n\,b^7 - 4\,a^{13}\,b\,b2 \\
&- 12\,b^3\,a^{11}\,b2 - 130\,b^4\,a^9\,b2 - 158\,b^5\,a^7\,b2 + 14\,a^2\,b2\,b^5 - 4\,b^4\,a^{10}\,b2 \\
&- 36\,b^5\,a^8\,b2 - 50\,n\,b^6\,a^3 + 10\,n\,b^6\,a^2 - 163\,n\,b^5\,a^5 + 38\,n\,b^7\,a + 4\,n\,a^{12}\,b^2 \\
&+ 2\,n\,a^{11}\,b^2 + 8\,n\,a^{14}\,b + 34\,n\,a^{10}\,b^3 + 40\,n\,a^9\,b^3 + 97\,n\,b^4\,a^8 - 10\,n\,b^4\,a^7 \\
&+ 134\,n\,b^4\,a^9 - 7\,n\,b^3\,a^{11} - 71\,n\,b^5\,a^7 - 279\,n\,b^5\,a^6 + 128\,n\,b^5\,a^8 + 26\,n\,b^4\,a^{10} \\
&- 312\,n\,b^6\,a^6 - 544\,n\,b^6\,a^5 - 332\,n\,b^6\,a^4 - 282\,n\,b^7\,a^4 - 168\,n\,b^7\,a^3 + 2\,n\,b^7\,a^2 \\
&+ 10\,n\,b^3\,a^8 - 14\,n\,b^4\,a^6 - 22\,n\,b^5\,a^4 + 8\,n\,a^{13}\,b^2 - 4\,n\,a^{12}\,b^3 + 56\,n\,b^5\,a^9 \\
&- 10\,n\,b^4\,a^{11} - 14\,n\,b^6\,a^7 + 26\,n\,b^6\,a^8 - 4\,n\,b^5\,a^{10} - 42\,n\,b^7\,a^6 - 182\,n\,b^7\,a^5 \\
&+ 28\,n\,b^5\,a^8\,b2 - 8\,n\,b^4\,a^{10}\,b2 - 10\,n\,b^5\,a^2\,b2 + 4\,n\,a^{12}\,b2\,b^2 + 12\,n\,a^{13}\,b2\,b \\
&+ 24\,n\,a^{12}\,b2\,b - 12\,n\,b^3\,a^{11}\,b2 + 36\,n\,b^4\,a^9\,b2 + 126\,n\,b^5\,a^7\,b2 + 2\,a^{14} + 2\,a^{15} \\
&+ 9\,a^{13}\,b + 2\,n\,b^2\,a^{10} + 4\,a^8\,n\,b2\,b^2 + 9\,n\,a^{13}\,b + 2\,n\,a^{12}\,b + 15\,a^{11}\,b\,n\,b2 \\
&+ 2\,a^{10}\,b\,n\,b2 + 2\,a^{15}\,n + 4\,a^{14}\,b2\,n + 6\,a^{13}\,b2\,n + 2\,a^{12}\,b2\,n + 2\,a^{14}\,n)/(2\,a^4 \\
&(a^2 + b\,a + b)^3(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 \\
&+ a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3)(a^2 - b)\,(1 + n))
\end{aligned}
$$

```
> D_4:=simplify(diff(CI, b_i, b_j, b_k));
```

$$D\_4 := -3(2\,b\,a^7\,b2 + 2\,b^2\,a^5\,b2 + 122\,b^2\,a^7\,b2 + 32\,b^2\,a^6\,b2 + 44\,b\,a^9\,b2 + 56\,b^3\,a^5\,b2 - 8\,b^3\,a^4\,b2 - 4\,a^3\,b2\,b^3 + 18\,a^8\,b2\,b - 8\,a^4\,b^4 - 16\,b^3\,a^6 - 12\,b^2\,a^8 - 72\,b^4\,a^5 - 16\,b^5\,a^3 - 92\,a^7\,b^3 + 2\,a^{10}\,b2 - 2\,n\,b^2\,a^5\,b2 - 2\,n\,a^7\,b2\,b + 8\,n\,b^3\,a^9\,b2 - 20\,n\,b^3\,a^8\,b2 - 16\,n\,b^4\,a^7\,b2 - 56\,n\,b^4\,a^6\,b2 - 112\,n\,b^3\,a^7\,b2 + 20\,n\,b^2\,a^{10}\,b2 - 64\,n\,b^4\,a^5\,b2 - 132\,n\,b^3\,a^6\,b2 + 10\,n\,b^2\,a^9\,b2 - 16\,n\,b^4\,a^4\,b2 + 64\,b^2\,a^9\,b2 + 172\,b^3\,a^7\,b2 + 176\,b^3\,a^6\,b2 + 156\,b^2\,a^8\,b2 + 64\,b^4\,a^5\,b2 + 16\,b^4\,a^4\,b2 - 16\,b^4\,a^3\,b2 + 28\,b\,a^{10}\,b2 + 56\,b^4\,a^6\,b2 + 56\,b^3\,a^8\,b2 - 8\,a^2\,b2\,b^4 + 16\,b^4\,a^7\,b2 - 178\,b^4\,a^6 - 135\,b^3\,a^8 - 64\,b^5\,a^4 - 9\,b^2\,a^{10} - a^{10}\,b + 4\,a^{11}\,b2 - 36\,b^2\,a^9 - 100\,b^5\,a^5 - 172\,a^7\,b^4 - 46\,b^3\,a^9 + 18\,b^2\,a^{11} - 54\,b^4\,a^8 + a^{12} + 14\,b^3\,a^{10} - 72\,b^5\,a^6 - 20\,a^7\,b^5 + 4\,a^9\,b^4 + 4\,a^{11}\,b + 10\,a^{12}\,b + 6\,n\,b^3\,a^5 + 2\,n\,b^4\,a^3 - 2\,n\,a^7\,b^2 + 88\,n\,b^6\,a^3 + 24\,n\,b^6\,a^2 + 160\,n\,b^5\,a^5 + 18\,n\,a^{11}\,b^2 + 14\,n\,a^{10}\,b^3 - 46\,n\,a^9\,b^3 - 54\,n\,b^4\,a^8 - 70\,n\,b^4\,a^7 + 4\,n\,b^4\,a^9 - 20\,n\,b^5\,a^7 + 16\,n\,b^5\,a^6 + 24\,n\,b^6\,a^5 + 80\,n\,b^6\,a^4 - 99\,n\,b^3\,a^8 + 70\,n\,b^4\,a^6 + 192\,n\,b^5\,a^4 - 9\,a^8\,b\,n\,b2 - 76\,a^7\,n\,b2\,b^2 - 26\,a^6\,n\,b2\,b^2 - 44\,a^5\,n\,b^3\,b2 + 8\,a^4\,n\,b^3\,b2 + 4\,a^3\,n\,b^3\,b2 - 8\,n\,b^6 + 42\,a^4\,n\,b^4 + 64\,a^3\,n\,b^5 - 8\,a^2\,n\,b^5 - 4\,a\,n\,b^5 - 2\,a^9\,n\,b - 21\,a^8\,n\,b^2 - 29\,a^7\,n\,b^3 + 17\,a^6\,n\,b^3 + 122\,a^5\,n\,b^4 + 2\,a^{13} - 9\,n\,b^2\,a^{10} - 62\,a^8\,n\,b2\,b^2 - 42\,n\,b^2\,a^9 + 8\,a^2\,n\,b^4\,b2 + 10\,n\,a^{12}\,b + 16\,a^{11}\,b\,n\,b2 + 18\,a^{10}\,b\,n\,b2 + 4\,a^{11}\,b\,n - 5\,a^{10}\,b\,n + 4\,a^{12}\,b2\,n + 2\,a^{13}\,n + a^{12}\,n - 16\,n\,b^6\,a - 5\,a^9\,b\,n\,b2 + 16\,n\,b^4\,a^3\,b2 + 4\,a^{11}\,b2\,n + a^{10}\,b2\,n)/(2\,a\,(a^2 + b\,a + b)^3(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3)(a^2 - b)\,(1 + n))$$

```
> D_5:=simplify(diff(CI, a_jk, y_i));
```

$$D\_5 := (2\,a^9\,b2 + 2\,a^8\,b2 + 2\,a^8\,b2\,b + 7\,b\,a^7\,b2 + 19\,b^2\,a^6\,b2 + 4\,b^2\,a^6 - 2\,b\,a^6\,b2 + 24\,b^2\,a^5\,b2 + 12\,b^3\,a^5\,b2 - 3\,b^3\,a^5 + 6\,b^2\,a^4\,b2 + 21\,b^3\,a^4\,b2 - 21\,a^4\,b^4 - 4\,a^4\,b^3 - 30\,b^4\,a^3 + 3\,a^3\,b2\,b^3 - 12\,b^5\,a^3 - 6\,a^2\,b2\,b^3 - 21\,b^5\,a^2 - 4\,b^4\,a^2 - 3\,b^5\,a + 4\,b^5)/(2\,a^6\,(1 + n)(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3))$$

```
> D_6:=simplify(diff(CI, a_jk, b_i));
```

$$D\_6 := (2\,a^8\,b2 + 2\,b\,a^7\,b2 - a^7\,b2 - 2\,b\,a^6 - 11\,b\,a^6\,b2 - 17\,a^5\,b2\,b - 8\,b^2\,a^5\,b2 + 3\,b^2\,a^5 - 4\,a^4\,b2\,b + 13\,a^4\,b^3 - 12\,b^2\,a^4\,b2 + 2\,a^4\,b^2 + 6\,b^4\,a^3 + 15\,b^3\,a^3 + 8\,b^4\,a^2 + 4\,a^2\,b2\,b^2 + 2\,b^3\,a^2 - 2\,b^4)/(2\,a^4\,(1 + n)(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3))$$

The spray coefficuents G^i of F is related to the spray coefficients bar{G}^i of alpha by

G^i = bar{G}^i + P y^i + Q^i

P = Xi * s_0 + Theta * r_00

Q^i = Q s^i_0 + (Phi * s_0 + Psi * r_00) * b^i

```
> Xi:=-eta*lambda*a*fb/fa:
> Theta:=(fab+fa*fb/f-eta*lambda*(fab*b+fbb*b2*a))/(2*fa):
> Q:=a*fb/fa:
> Phi:=-(tau+eta*lambda^2)*fb*a/fa:
> Psi1:=(fbb*a-(tau+eta*lambda^2)*(fab*b+fbb*b2*a))/(2*fa):
```

```
> Xi:=simplify(Xi);
```

$$\Xi := -a^2\,(-3\,b\,a^3 - b^2\,a - 2\,b\,a^2 + 2\,b^2 + a^5 + b\,a^4 - 3\,b^2\,a^2)\,(a+1)/((a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3)(a^2 - b))$$

```
> Theta:=simplify(Theta);
```

$$\Theta := a(-2\,a^3\,b2\,b - 7\,a^4\,b2\,b + 2\,b\,a^6\,b2 - 3\,a^5\,b2\,b - 4\,b^2\,a^4\,b2 - 6\,a^3\,b2\,b^2 + 2\,b^3\,a - 2\,b\,a^5 - 5\,a^4\,b^2 + 5\,b^3\,a^2 - 2\,b\,a^6 + 4\,b^4\,a + b^4 - 6\,b^2\,a^5 - 4\,a^4\,b^3 + 3\,b^4\,a^2 + 2\,a^7\,b + b^2\,a^6 + 2\,a^7\,b2 + a^8 + a^6\,b2 + 2\,a\,b2\,b^2)/(2\,(a^2 + b\,a + b)(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3)(a^2 - b))$$

```
> Q:=simplify(Q);
```

$$Q := \frac{a^2\,(a+1)}{a^2 - b}$$

```
> Phi:=simplify(Phi);
```

$$\Phi := -(a+1)\,(2\,a^3 + 2\,b\,a^2 + a^2 + b\,a - b)\,a^4/((a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3)(a^2 - b))$$

```
> Psi1:=simplify(Psi1);
```

$$\Psi 1 := b\,(2\,a^3 + 2\,b\,a^2 + a^2 + b\,a - b)\,a^2/(2\,(a^2 - b)(a^6 + 2\,a^5\,b2 + 2\,b\,a^5 - b\,a^4 + a^4\,b2 + 2\,a^4\,b2\,b + a^4\,b^2 - 4\,b^2\,a^3 + a^3\,b2\,b - a^2\,b2\,b - 3\,b^3\,a^2 - b^2\,a^2 - b^3\,a + b^3))$$